High School Mathematics Contest
The departments of
MATHEMATICS and MATHEMATICS EDUCATION
EAST CAROLINA UNIVERSITY

COMPREHENSIVE: 2011

SOLUTIONS TO SELECT QUESTIONS

(N = 178)

1. If $\sin x - \cos x = \dfrac{\sqrt{3}}{2}$, then $(\sin x)(\cos x) =$

(A) 1 (B) $\dfrac{\sqrt{3}}{4}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{1}{4}$ (E) $\dfrac{1}{8}$

Correct Answer: (E)
Answer Distribution: (A) 10.1% (B) 30.9% (C) 28.1% (D) 13.9% (E) 12.4% (Other) 5.1%

Solution:

$$(\sin x - \cos x)^2 = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\rightarrow \sin^2 x - 2\sin x\cos x + \cos^2 x = \frac{3}{4}$$

noting, $\sin^2 x + \cos^2 x = 1$

$$\rightarrow \sin x\cos x = \frac{1}{8}$$

2. The determinant $\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} =$

(A) 0 (B) 1 (C) -1 (D) $a+b+c$ (E) $(a+b)(b+c)(a+c)$

Correct Answer: (A)
Answer Distribution: (A) 10.7% (B) 18.0% (C) 17.4% (D) 22.2% (E) 30.9% (Other) 2.8%

Solution:

$$\begin{aligned} &\det\begin{bmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{bmatrix} \\ &= \det\begin{bmatrix} b & a+c \\ c & a+b \end{bmatrix} - \det\begin{bmatrix} a & b+c \\ c & a+b \end{bmatrix} + \det\begin{bmatrix} a & b+c \\ b & a+c \end{bmatrix} \\ &= b(a+b)-c(a+c)-a(a+b)+c(b+c)+a(a+c)-b(b+c) \\ &= 0 \end{aligned}$$

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3. The prime factorization of 168,750 is $(2)^2(3)^3(5)^5$. How many factors of 168,750 are there?

(A) 72 (B) 60 (C) 38 (D) 30 (E) 10

Correct Answer: (A)
Answer Distribution: (A) 7.3% (B) 12.4% (C) 16.9% (D) 26.4% (E) 33.1% (Other) 3.9%

Solution:

$$(2+1)(3+1)(5+1)=72$$

Any factor will be of the form $(2)^x(3)^y(5)^z$, where x can be 0,1,2 (2+1 choices), where y can be 0,1,2,3 (3+1 choices), and z can be 0,1,2,3,4,5 (5+1) choices.

4. Which of the following are fourth roots of $-8+8i\sqrt{3}$?

I. $\sqrt{3}+i$ II. $-\sqrt{3}-i$ III. $1-i\sqrt{3}$

(A) I Only (B) II only (C) III only (D) I & III only (E) all 3

Correct Answer: (E)
Answer Distribution: (A) 10.7% (B) 14.6% (C) 32.6% (D) 19.7% (E) 15.2% (Other) 7.3%

Solution:
(I). $\sqrt{3}+i$ squared yields $2+2i\sqrt{3}$ and squared again yields $-8+8i\sqrt{3}$.
(II). $-\sqrt{3}-i$ is simply the negative of (I) and thus will have the same 4^{th} power.
(III). $1-i\sqrt{3}$ squared yields $-2-2i\sqrt{3}$ which is negative of the square of (I).
Hence, all have same 4^{th} power.

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5. If $a+b=1$ and $a^2+b^2=2$, then $a^3+b^3=$

(A) $2\left(\frac{1-\sqrt{3}}{2}\right)^3$ (B) 2 (C) 2.5 (D) 3 (E) $2\left(\frac{1+\sqrt{3}}{2}\right)^3$

Correct Answer: (C)
Answer Distribution: (A) 7.3% (B) 9.0% (C) 14.0% (D) 38.8% (E) 23.0% (Other) 7.9%

Solution:

$$\begin{aligned}
&(a+b)^2 = 1^2 \\
&\to a^2+2ab+b^2=1 \\
&\to (a^2+b^2)+2ab=1 \\
&\to 2+2ab=1 \\
&\to ab=-0.5
\end{aligned}$$

$$\begin{aligned}
&(a+b)^3 = 1^3 \\
&\to a^3+3a^2b+3ab^2+b^3=1 \\
&\to (a^3+b^3)+3ab(a+b)=1 \\
&\to (a^3+b^3)+3(-0.5)(1)=1 \\
&\to a^3+b^3=2.5
\end{aligned}$$